Jack Ganssle believes that embedded development can be much more efficient than it usually is, and that we can – and must – create more reliable products. He conducts one-day seminars that teach ways to produce better firmware faster. For more information see www.ganssle.com.
Floating Point Approximations

Most embedded processors don’t know how to compute trig and other complex functions. Programming in C we’re content to call a library routine that does all of the work for us. Unhappily this optimistic approach often fails in real time systems where size, speed and accuracy are all important issues.

The compiler’s runtime package is a one-size-fits-all proposition. It gives a reasonable trade-off of speed and precision. But every embedded system is different, with different requirements. In some cases it makes sense to write our own approximation routines. Why?

*Speed* – Many compilers have very slow runtime packages. *A clever approximation may eliminate the need to use a faster CPU.*

*Predictability* – Compiler functions vary greatly in execution time depending on the input argument. Real time systems must be *predictable* in the time domain. The alternative is to always assume worst case execution time, which again may mean your CPU is too slow, too loaded, for the application.

*Accuracy* – Read the compiler’s manuals carefully! Some explicitly do not support the ASNI C standard, which requires all trig to be double precision. (8051 compilers are notorious for this). Alternatively, why pay the cost (in time) for double precision math when you only need 5 digits of accuracy?

*Size* – When memory is scarce, using one of these approximations may save much code space. If you only need a simple cosine, why include the entire floating point trig library?

This collection is not an encyclopedia of all possible approximations; rather, it’s the most practical ones distilled from the bible of the subject, *Computer Approximations* by John Hart (ISBN 0-88275-642-7). Unfortunately this work is now out of print. It’s also very difficult to use without a rigorous mathematical background.

All of the approximations here are polynomials, or ratios of polynomials. All use very cryptic coefficients derived from Chebyshev series and Bessel functions. Rest assured that these numbers give minimum errors for the indicated ranges. Each approximation (with a few exceptions) has an error chart so you can see exactly where the errors occur. In some cases, if you’re using a limited range of input data, your accuracies can far exceed the indicated values. For instance, *cos_73* is accurate to 7.3 decimal digits over the 0 to 360 degree range. But as the graph shows, in the range 0 to 30 degrees you can get almost an order of magnitude improvement.
Do be wary of the precision of your compiler’s floating point package. Some treat doubles as floats. Others, especially for tiny CPUs like the PIC, cheat and offer less than full 32 bit floating point precision.

All of the code for the following approximations was compiled with Microsoft’s Visual C++ 6.0. The source is available at [www.ganssle.com/approx/sincos.cpp](http://www.ganssle.com/approx/sincos.cpp). It includes test code that writes a text file of results and errors; imported to a spreadsheet we can see just how accurate they are.

**General Trig Notes**

We generally work in radians rather than degrees. The 360 degrees in a circle are equivalent to $2\pi$ radians; thus, one radian is $360/(2\pi)$, or about 57.3 degrees. This may seem a bit odd till you think of the circle’s circumference, which is $2\pi r$; if $r$ (the circle’s radius) is one, the circumference is indeed $2\pi$.

The conversions between radians and degrees are:

\[
\text{Angle in radians} = \text{angle in degrees} \times \frac{2\pi}{360} \\
\text{Angle in degrees} = \text{angle in radians} \times \frac{360}{2\pi}
\]

<table>
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<th>Tangent</th>
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</tbody>
</table>

**Cosine and Sine**

The following examples all approximate the cosine function; sine is derived from cosine via the relationship:

\[
\sin(x) = \cos(\pi/2 - x)
\]

In other words, the sine and cosine are the same function, merely shifted 90º in phase. The sine code is (assuming we’re calling cos_32, the lowest accuracy cosine approximation):
All of the cosine approximations in this chapter compute the cosine accurately over the range of 0 to \(\frac{\pi}{2}\) (0 to 90º). That surely denies us of most of the circle! Approximations in general work best over rather limited ranges; it's up to us to reduce the input range to something the approximation can handle accurately.

Therefore, before calling any of the following cosine approximations we assume the range has been reduced to 0 to \(\frac{\pi}{2}\) using the following code:

```c
// Math constants
double const pi=3.1415926535897932384626433; // pi
double const twopi=2.0*pi; // pi times 2
double const halfpi=pi/2.0; // pi divided by 2

// This is the main cosine approximation "driver"
// It reduces the input argument's range to [0, pi/2],
// and then calls the approximator.

float cos_32(float x){
    int quad;    // what quadrant are we in?
    x=fmod(x, twopi);   // Get rid of values > 2*pi
    if(x<0)x=-x;   // cos(-x) = cos(x)
    quad=int(x/halfpi);  // Get quadrant # (0 to 3)  switch (quad){
        case 0: return  cos_32s(x);
        case 1: return -cos_32s(pi-x);
        case 2: return -cos_32s(x-pi);
        case 3: return  cos_32s(twopi-x);
    }
}
```

This code is configured to call \texttt{cos\_32s}, which is the approximation (detailed shortly) for computing the cosine to 3.2 digits accuracy. Use this same code, though, for all cosine approximations; change \texttt{cos\_32s} to \texttt{cos\_52s, cos\_73s or cos\_121s}, depending on which level of accuracy you need. See the complete listing for a comprehensive example.

If you can guarantee that the input argument will be greater than zero and less than 2 \(\pi\), delete the two red lines in the listing above to get even faster execution.

Be clever about declaring variables and constants. Clearly, working with the \texttt{cos\_32} approximation nothing must be declared “\texttt{double}”. Use \texttt{float} for more efficient code.

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Reading the complete listing you’ll notice that for \( \text{cos}_\text{32} \) and \( \text{cos}_\text{52} \) we used floats everywhere; the more accurate approximations declare things as doubles.

One trick that will speed up the approximations is to compute \( x^2 \) by incrementing the characteristic of the floating point representation of \( x \). You’ll have to know exactly how the numbers are stored, but can save hundreds of microseconds over performing the much clearer “\( x \times x \)” operation.

How does the range reduction work? Note that the code divides the input argument into one of four “quadrants” – the very same quadrants of the circle shown below:

\[ \begin{array}{c|c|c}
0 & 1 & 2 \\
\hline
\pi/2 & 0 & 3 \\
\pi & 2 & \pi/2 \\
3\pi/2 & 1 & 0
\end{array} \]

*Quadrants 0 to 3 of the circle*

- For the first quadrant (0 to \( \pi/2 \)) there’s nothing to do since the cosine approximations are valid over this range.
- In quadrant 1 the cosine is symmetrical with quadrant 0, if we reduce it’s range by subtracting the argument from \( \pi \). The cosine, though, is negative for quadrants 1 and 2 so we compute \( -\cos(\pi - x) \).
- Quadrant 2 is similar to 1.
- Finally, in 3 the cosine goes positive again; if we subtract the argument from 2 \( \pi \) it translates back to something between 0 and \( \pi/2 \).

The approximations do convert the basic polynomial to a simpler, much less computationally expensive form, as described in the comments. All floating point operations take appreciable amounts of time, so it’s important to optimize the design.
cos_32 computes a cosine to about 3.2 decimal digits of accuracy. Use the range reduction code (listed earlier) if the range exceeds 0 to $\pi/2$. The plotted errors are absolute (not percent error).
cos_52 computes a cosine to about 5.2 decimal digits of accuracy. Use the range reduction code (listed earlier) if the range exceeds 0 to π/2. The plotted errors are absolute (not percent error).
cos_73 computes a cosine to about 7.3 decimal digits of accuracy. Use the range
reduction code (listed earlier) if the range exceeds 0 to π/2. Also plan on using double
precision math for the range reduction code to avoid losing accuracy. The plotted errors
are absolute (not percent error).

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cos\_121 computes a cosine to about 12.1 decimal digits of accuracy. Use the range reduction code (listed earlier) if the range exceeds 0 to π/2. Also plan on using double
precision math for the range reduction code to avoid losing accuracy. The plotted errors are absolute (not percent error).

**Higher Precision Cosines**

Given a large enough polynomial there’s no limit to the possible accuracy. A few more algorithms are listed here. These are all valid for the range of 0 to $\pi/2$, and all can use the previous range reduction algorithm to change any angle into one within this range. All take an input argument in radians.

No graphs are included because these exceed the accuracy of the typical compiler’s built-in cosine function… so there’s nothing to plot the data against.

Note that C’s `double` type on most computers carries about 15 digits of precision. So for these algorithms, especially for the 20.2 and 23.1 digit versions, you’ll need to use a data type that offers more bits. Some C’s support a `long double`. But check the manual carefully! Microsoft’s Visual C++, for instance, while it does support the `long double` keyword, converts all of these to `double`.

Accurate to about 14.7 decimal digits over the range $[0, \pi/2]$:

\[
\begin{align*}
c_1 &= 0.99999999999999806767 \\
c_2 &= -0.499999999999896568 \\
c_3 &= 0.041666666666581174292 \\
c_4 &= -0.0013888888613613522 \\
c_5 &= 0.0000024801582876042427 \\
c_6 &= -0.0000002755693576863181 \\
c_7 &= 0.0000000020858327958707 \\
c_8 &= -0.00000000011080716368 \\
\end{align*}
\]

\[
\cos(x) = c_1 + x^2(c_2 + x^2(c_3 + x^2(c_4 + x^2(c_5 + x^2(c_6 + x^2(c_7 + x^2c_8))))))
\]

Accurate to about 20.2 decimal digits over the range $[0, \pi/2]$:

\[
\begin{align*}
c_1 &= 0.9999999999999999999936329 \\
c_2 &= -0.49999999999999999948362843 \\
c_3 &= 0.04166666666665975670054 \\
c_4 &= -0.001388888888885302082298 \\
c_5 &= 0.0000024801587301492746422297 \\
c_6 &= -0.00000027557319209666748555 \\
c_7 &= 0.0000000020876755667423458605 \\
c_8 &= -0.000000000114706701991777771 \\
c_9 &= 0.0000000000000477687298095717 \\
c_{10} &= -0.000000000000015119893746887 \\
\end{align*}
\]

\[
\cos(x) = c_1 + x^2(c_2 + x^2(c_3 + x^2(c_4 + x^2(c_5 + x^2(c_6 + x^2(c_7 + x^2(c_8 + x^2(c_9 + x^2c_{10})))))))))
\]

Accurate to about 23.1 decimal digits over the range $[0, \pi/2]$:

\[
\begin{align*}
c_1 &= 0.9999999999999999999999914771 \\
c_2 &= -0.4999999999999999999991637437 \\
c_3 &= 0.0416666666666665319411988 \\
c_4 &= -0.0013888888888880310186415 \\
c_5 &= 0.000002480158730158702330045157 \\
c_6 &= -0.000000275573192239332256421489 \\
\end{align*}
\]
c7 = 0.000000002087675698165412591559
\n\nc8 = -0.000000000114707451267755432394
\n\nc9 = 0.0000000000477945439406649917
\n\nc10 = -0.000000000000015612263428827781
\n\nc11 = 0.0000000000000039912654507924

\cos(x) = c1 + x^2(c2 + x^2(c3 + x^2(c4 + x^2(c5 + x^2(c6 +
\ns^2(c7 + x^2(c8 + x^2(c9 + x^2(c10 + x^2*c11)))))))))))
Tangent

The tangent of an angle is defined as $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Unhappily this is not the best choice, though, for doing an approximation. As $\cos(x)$ approaches zero the errors propagate rapidly. Further, at some points like $\pi/4$ (see the previous graphs of sine and cosine errors) the errors of sine and cosine reinforce each other; both are large and have the same sign.

So we’re best off using a separate approximation for the tangent. All of the approximations we’ll use generate a valid tangent for angles in the range of 0 to $\pi/4$ (0 to 45 degrees), so once again a range reduction function will translate angles to this set of values.

```c
//
//  This is the main tangent approximation "driver"
//  It reduces the input argument's range to [0, pi/4],
//  and then calls the approximator.
// Enter with positive angles only.
//
// WARNING: We do not test for the tangent approaching infinity,
// which it will at x=pi/2 and x=3*pi/2. If this is a problem
// in your application, take appropriate action.
//
float tan_32(float x){
  int octant;     // what octant are we in?
  x=fmod(x, twopi);    // Get rid of values >2 *pi
  octant=int(x/qtrpi);   // Get octant # (0 to 7)
  switch (octant){
    case 0: return      tan_32s(x              *four_over_pi);
    case 1: return  1.0/tan_32s((halfpi-x)     *four_over_pi);
    case 2: return -1.0/tan_32s((x-halfpi)     *four_over_pi);
    case 3: return -    tan_32s((pi-x)         *four_over_pi);
    case 4: return      tan_32s((x-pi)         *four_over_pi);
    case 5: return  1.0/tan_32s((threehalfpi-x)*four_over_pi);
    case 6: return -1.0/tan_32s((x-threehalfpi)*four_over_pi);
    case 7: return -    tan_32s((twopi-x)      *four_over_pi);
  }
}
```

The code above does the range reduction and then calls `tan_32`. When using the higher precision approximations substitute the appropriate function name for `tan_32`.

The reduction works much like that for cosine, except that it divides the circle into octants and proceeds from there. One quirk is that the argument is multiplied by $4/\pi$. This is because the approximations themselves actually solve for $\tan((\pi/4)x)$.

The listings that follow give the algorithms needed.

Remember that $\tan(90)$ and $\tan(270)$ both equal infinity. As the input argument gets close to 90 or 270 the value of the tangent skyrocket, as illustrated on the following error charts. Never take a tangent close to 90 or 270 degrees!
tan_32 computes the tangent of \( \pi/4 \times x \) to about 3.2 digits of accuracy. Use the range reduction code to translate the argument to 0 to \( \pi/4 \), and of course to compensate for the peculiar \( \pi/4 \) bias required by this routine. Note that the graphed errors are percentage error, not absolute.

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tan_56 computes the tangent of $\pi/4 \times x$ to about 5.6 digits of accuracy. Use the range reduction code to translate the argument to 0 to $\pi/4$, and of course to compensate for the peculiar “$\pi/4$” bias required by this routine. Note that the graphed errors are percentage error, not absolute.

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tan_82 computes the tangent of $\pi/4 \times x$ to about 8.2 digits of accuracy. Use the range reduction code to translate the argument to 0 to $\pi/4$, and of course to compensate for the peculiar "$\pi/4$" bias required by this routine. Note that variables are declared as “double”. The graphed errors are percentage error, not absolute.

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tan_14 computes the tangent of $\pi/4 \cdot x$ to about 14.1 digits of accuracy. Use the range reduction code to translate the argument to 0 to $\pi/4$, and of course to compensate for the peculiar “$\pi/4$” bias required by this routine. Note that variables are declared as “double”. The graphed errors are percentage error, not absolute.
Higher Precision Tangents

Given a large enough polynomial there’s no limit to the possible accuracy. A few more algorithms are listed here. These are all valid for the range of 0 to $\pi/4$, and all should use the previous range reduction algorithm to change any angle into one within this range. All take an input argument in radians, though it is expected to be mangled by the $\pi/4$ factor. The prior range reducer will correct for this.

No graphs are included because these exceed the accuracy of the typical compiler’s built-in cosine function… so there’s nothing to plot the data against.

Note that C’s `double` type on most computers carries about 15 digits of precision. So for these algorithms, especially for the 20.2 and 23.1 digit versions, you’ll need to use a data type that offers more bits. Some C’s support a `long double`. But check the manual carefully! Microsoft’s Visual C++, for instance, while it does support the `long double` keyword, converts all of these to `double`.

Accurate to about 20.3 digits over the range of 0 to $\pi/4$:
\[
\tan(x\pi/4) = x(c_1 + x^2(c_2 + x^2(c_3 + x^2(c_4))))
\]
\[
/ (c_5 + x^2(c_6 + x^2(c_7 + x^2(c_8 + x^2))))
\]

```
c1= 10881241.46289544215469695742
C2=  895306.0870564145957447087575
C3= 14181.99563014366386894487566
C4=  45.63638305432707847378129653
C5= 13854426.92637036839270054048
C6=  3988641.468163077300701338784
C7= 135299.474455002368086759195
C8=  1014.19757617656429288596025
```

Accurate to about 23.6 digits over the range of 0 to $\pi/4$:
\[
\tan(x\pi/4) = x(c_1 + x^2(c_2 + x^2(c_3 + x^2(c_4 + x^2(c_5)))))
\]
\[
/ (c_6 + x^2(c_7 + x^2(c_8 + x^2)))
\]

```
c1= 4130240.588996024013440146267
C2=  349781.8562517381616631012487
C3=  6170.3175814249424531944348
C4=  27.94920941380194872760036319
C5=  0.0175143807040383602666563058
C6=  5258785.64717998252478541780825
C7=-1526650.549072940686776259893
C8=  54962.5161606290536115230566
C9=  497.495460280917265024506937
```

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Arctangent, arcsine and arccosine

The arctangent is the same as the inverse tangent, so \( \arctan(\tan(x)) = x \). It’s often denoted as “\( \text{atan}(x) \)” or “\( \tan^{-1}(x) \)”. 

In practice the approximations for inverse sine and cosine aren’t too useful; mostly we derive these from the arctangent as follows:

\[
\text{Arcsine}(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) \\
\text{Arccosine}(x) = \frac{\pi}{2} - \arcsine(x) \\
= \frac{\pi}{2} - \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)
\]

The approximations are valid for the range of 0 to \( \pi/12 \). The following code, based on that by Jack Crenshaw in his *Math Toolkit for Real-Time Programming*, reduces the range appropriately:

```c
// This is the main arctangent approximation "driver"
// It reduces the input argument's range to [0, pi/12],
// and then calls the approximator.

double atan_66(double x){
    double y;     // return from atan__s function
    int complement= FALSE;    // true if arg was >1
    int region= FALSE;    // true depending on region arg is in
    int sign= FALSE;    // true if arg was < 0

    if (x <0 ){
        x=-x;
        sign=TRUE;    // arctan(-x)=-arctan(x)
    }
    if (x > 1.0){
        x=1.0/x;    // keep arg between 0 and 1
        complement=TRUE;
    }
    if (x > tantwelfthpi){
        x = (x-tansixthpi)/(1+tansixthpi*x); // reduce arg to under tan(pi/12)
        region=TRUE;
    }

    y=atan_66s(x);     // run the approximation

    if (region) y+=sixthpi;   // correct for region we're in
    if (complement)y=halfpi-y;   // correct for 1/x if we did that
    if (sign)y=-y;     // correct for negative arg
    return (y);
```
atan_66 computes the arctangent to about 6.6 decimal digits of accuracy using a simple rational polynomial. It’s input range is 0 to π/12; use the previous range reduction code.
atan_137 computes the arctangent to about 13.7 decimal digits of accuracy using a simple rational polynomial. It’s input range is 0 to \( \pi /12 \); use the previous range reduction code.